1. Let $\mathbf{L}$, $\mathbf{A}$ be some fixed vectors in $\mathbf{R}^3$, and $\mu = \mathbf{L}\cdot\mathbf{A}$. For vector variable $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$, let us consider the system of equations

$$\mathbf{L}\cdot\mathbf{x} = \mu r, \quad r - \mathbf{A}\cdot\mathbf{x} = L^2 - \mu^2$$

where r is the length of $\mathbf{x}$, and L is the length of $\mathbf{L}$. Assume that $\mathbf{L}$ is a none-zero vector.

(i) Show that this system of equations determines a conic.

(ii) Find the eccentricity of this conic in terms of $\mathbf{L}$ and $\mathbf{A}$.

2. On the Lorentzian vector space $\mathbf{R}^{1,3} := (\mathbf{R}^4, \cdot)$, the Lorentzian dot product $\cdot$ is given by formula

$$[x_0, x_1, x_2, x_3]^T \cdot [y_0, y_1, y_2, y_3]^T = x_0y_0 - x_1y_1 - x_2y_2 - x_3y_3.$$

Show that, if e is a time-like unit vector (i.e., $e\cdot e = 1$), then formula

$$\langle x, y\rangle := 2(x\cdot e)(y\cdot e) - x\cdot y$$

defines an inner product on $\mathbf{R^4}$. In particular, if $e = e_0 := [1, 0, 0, 0]^T$, we have

$$\left\langle [x_0, x_1, x_2, x_3]^T \cdot, [y_0, y_1, y_2, y_3]^T \right\rangle = x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3.$$

3. Let l, a be some fixed 4-dimensional Lorentz vectors such that $l\cdot l = -1$, $l\cdot a = 0$, and $a_0 > 0$. Here a_0 denotes the temporal component of a.

(i) Show that the intersection of the plane

$$l\cdot x = 0, \quad a\cdot x = 1$$

with the future light cone

$$x\cdot x = 0, \quad x_0 > 0$$

is a conic.

(ii) Show that this conic is an eclipse, a parabola, or a branch of a hyperbola according as $a\cdot a$ is positive, zero and negative.

4. Let $\mathrm{H}_n(\mathbf{C})$ be the set of complex hermitian matrices of order n, $\mathbf{C}^n_*$ be the set of non-zero column matrices with n complex entries. Consider the map

$$q: \quad \mathbf{C}^n_* \to \mathrm{H}_n(\mathbf{C})$$

which maps $z \in \mathbf{C}^n_*$ to $\bar{z}^T z$. Here T stands for transpose and $\bar{z}$ is the complex conjugation of z.

(i) Show that the image of q, $\operatorname{Im} q$, is precisely the set of rank one, semi-positive hermitian matrices of order n. Let us denote this set by $\mathcal{C}_1$.

(ii) Let $\mathbf{C}P^k$ denote the set of 1-dimensional complex vector subspaces of the complex vector space $\mathbf{C}^{k+1}$. For matrix A, we use $\operatorname{tr} A$ to denote the trace of A and $\operatorname{Col} A$ to denote the column space of A. Show that the map

$$\mathcal{C}_1 \to (0,\infty)\times\mathbf{C}P^{n-1}$$

which maps $x\in\mathcal{C}_1$ to $(\operatorname{tr} x, \operatorname{Col} x)$ is a bijection.

(iii) Show that $\mathrm{H}_n(\mathbf{C})$ is a real vector space. (So it can be viewed as a real affine space with the same dimension.)

(iv) For any smooth map α: $I \to \mathrm{H}_n(\mathbf{C})$ where I is an open interval containing 0, if the image of α is inside $\mathcal{C}_1$, we say that α is a smooth parametrized curve on $\mathcal{C}_1$, passing through point $\alpha(0)$. Show that, for any $x \in \mathcal{C}_1$, the image of L_x (the Jordan multiplication by x), $\operatorname{Im} L_x$, can be described this way: $u \in \operatorname{Im} L_x$ if and only if $u = \alpha'(0)$ for some smooth parametrized curve α on $\mathcal{C}_1$, passing through point x. (In case you know that $\mathcal{C}_1$ is a smooth manifold, this proves that the tangent space of $\mathcal{C}_1$ at point x is $\{x\}\times\operatorname{Im} L_x$.)